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Infinite Hamiltonian paths in Cayley digraphs of hyperbolic symmetry groups

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Abstract

The hyperbolic symmetry groups $[p,q]$, $[p,q]^+$, and $[p^+,q]$ have certain natural generating sets. We determine whether or not the corresponding Cayley digraphs have one-way infinite or two-way infinite directed Hamiltonian paths. In addition, the analogous Cayley graphs are shown to have both one-way infinite and two-way infinite Hamiltonian paths.

1. Introduction

Many of M.C. Escher's well-known works of art are repeating patterns in the hyperbolic plane. A computer can easily create similar patterns: it can take a basic motif and reproduce it throughout the plane [2]. But the computer needs to be guided by a systematic method for visiting all the locations at which to copy the motif. For reasons of efficiency, no location should be visited more than once, so this systematic method corresponds to an infinite Hamiltonian path in a certain graph (or digraph). Because this digraph turns out to be a so-called Cayley digraph of the symmetry group of the pattern, it is of interest to study the existence of Hamiltonian paths in Cayley digraphs of hyperbolic symmetry groups. That is the topic of this paper.

Definition. Let Δ be a generating set for a (finite or infinite) group Γ. The *Cayley digraph* $\mathrm{Cay}(\Gamma;\Delta)$ (or simply $\mathrm{Cay}(\Gamma)$ or $\mathrm{Cay}(\Delta)$ if no confusion is likely to result) of

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Γ with generators Δ is a directed graph. Its vertices are the elements of Γ, and there is an arc from γ to $\gamma\delta$ for each $\gamma \in \Gamma$ and $\delta \in \Delta$.

Definition. In a countably infinite digraph G, there are two possible types of directed Hamiltonian paths: (1) A *one-way infinite directed Hamiltonian path* in G is a list $v_1, v_2, \ldots$ of the vertices of G such that there is an arc from v_i to v_{i+1} for each i. (2) A *two-way infinite directed Hamiltonian path* in G is a similar list $\ldots, v_{-1}, v_0, v_1, \ldots$.

We are interested in the hyperbolic symmetry groups $[p,q]$, $[p,q]^+$, and $[p^+,q]$ (see [1, Section 5]): For any natural numbers p and q with $(p-2)(q-2) > 4$, there is a discrete group

$$[p,q] = \langle T_p, T_q, T_2 \mid T_p^2 = T_q^2 = T_2^2 = (T_pT_q)^2 = (T_qT_2)^p = (T_pR_2)^q = e\rangle$$

of isometries of the hyperbolic plane. Its natural generating set, $\{T_p, T_q, T_2\}$, consists of the reflections in the sides of a (hyperbolic) right triangle of angles π/p, π/q, and $\pi/2$ (with T_k being the reflection in the side opposite the angle of π/k). It has an orientation-preserving subgroup $[p,q]^+$ of index 2 generated by the rotations $R = T_qT_2, S = T_2T_p$, and $T = T_pT_q$, with the presentation:

$$[p,q]^+ = \langle R, S, T \mid R^p = S^q = T^2 = RST = e\rangle.$$

Since $\{R, S, T\}$ is obviously *not* a minimal generating set, it is natural to consider the Cayley digraph of $[p,q]^+$ with generating set $\{R, S\}$, $\{R, S^{-1}\}$, or $\{R, T\}$. (One could also consider $\{S, T\}$, $\{R^{-1}, S\}$, or $\{R^{-1}, S^{-1}\}$ but, because of the symmetry in p and q, these are redundant.) If q is even, then $[p,q]$ has another subgroup $[p^+,q]$ of index 2, generated by the rotation $R = T_qT_2$ and the reflection T_p. The corresponding presentation is

$$[p^+,q] = \langle R, T_p \mid R^p = T_p^2 = (R^{-1}T_pRT_q)^{q/2} = e\rangle.$$

Remark. Each of the Cayley digraphs considered here is planar. For example Fig. 9 indicates how to embed Cay(R, S, T) in the (hyperbolic) plane.

Our results on directed Hamiltonian paths are summarized in Table 1. For each of the Cayley digraphs, we also show that there are both one-way infinite and two-way infinite Hamiltonian paths in the graph obtained by replacing all the arcs with undirected edges (see Section 8). For similar work on Euclidean symmetry groups, see [4]. Other papers on directed Hamiltonian paths in infinite Cayley digraphs are [3, 5, 6].

2. Preliminaries

Throughout the paper, p and q are natural numbers satisfying $(p-2)(q-2) > 4$.

Table 1
The types of infinite directed Hamiltonian paths that exist in the various Cayley digraphs, and references to the relevant results in the paper [one-way; two-way]

$\mathrm{Cay}([p,q]; T_p, T_q, T_2)$	both one-way and two-way [5.4, 3.2]
$\mathrm{Cay}([p,q]^+; R, S, T)$	both one-way and two-way [4.7, 6.3]
$\mathrm{Cay}([p,q]^+; R, S)$	one-way only (and this only if $p, q > 3$) [4.1, 7.1; 7.6]
$\mathrm{Cay}([p,q]^+; R, S^{-1})$	two-way only [7.3; 3.3]
$\mathrm{Cay}([p,q]^+; R, T)$	two-way only (and this only if $p > 3$) [7.2; 6.2, 7.4]
$\mathrm{Cay}([p^+,q]; R, T_p)$	neither [7.7; 7.5]

Definition. We use $\{p, q\}$ to denote the regular tessellation of the hyperbolic plane by regular p-gons, with q of the p-gons meeting at each vertex.

Note that $[p, q]$ is the symmetry group of $\{p, q\}$. The generators T_p, T_q, and T_2 are reflections in a side, in an inradius, and in a circumradius, respectively, of one of the p-gons of $\{p, q\}$.

Definition. A subset X of the hyperbolic plane is *convex* if, for every pair of points x and y in X, it contains the hyperbolic line segment $\overline{xy}$ joining x and y.

Caution. A convex subset of the hyperbolic plane need not appear convex (in the usual Euclidean sense) in the Poincaré disk model of hyperbolic geometry. For example, a hyperbolic triangle will not contain all the Euclidean line segments joining pairs of its vertices.

Definition. A *filtration* of the hyperbolic plane is an increasing chain $A_1 \subset A_2 \subset \cdots$ of (connected closed) subsets, whose union is the entire hyperbolic plane. We may call A_n the nth *filtration set.*

Definition 2.1. In this paper, filtrations arise from tessellations of the hyperbolic plane as follows. We let the first filtration set be the union of some of the polygons of the tessellation. Then the nth filtration set A_n is defined inductively to be the union of all those polygons of the tessellation that intersect A_{n-1}. (Note that $A_{n-1} \subset A_n$).

Definition. The term *graph* will always refer to an undirected graph, and the term *digraph* will always refer to a directed graph. We use the term *edge* to refer to the edges of a graph, and the term *arc* to refer to the directed edges of a digraph.

Definition. A *circuit* in a digraph G is a closed directed walk with no repeated vertices. That is, it is a sequence $v_1, \ldots, v_n$ of distinct vertices of G such that there is an arc from v_i to v_{i+1} for $i = 1, \ldots, n-1$, and from v_n to v_1. (Some authors call this a directed cycle.)

Definition. If a digraph is embedded in the plane, then in traversing any arc (from initial vertex to terminal vertex), one traverses a portion of the boundary of each of two regions. One boundary, that of the region to the *right* of the arc, is being traversed clockwise, and the other boundary, that of the region to the *left* of the arc, is being traversed counterclockwise.

3. Two-way infinite directed Hamiltonian paths in $\mathrm{Cay}([p,q])$ and $\mathrm{Cay}(R,S^{-1})$

Definition. An infinite graph G has *only one end* if, for every finite set A of vertices of G, there is only one infinite component in the subgraph $G\backslash A$ induced on the complement of A.

Theorem 3.1 (cf. [7, p. 56]). *Let G be a connected, countably infinite graph with only one end. Assume every vertex has finite valence. Suppose there is a set $\mathscr{C}$ of (pairwise disjoint) cycles containing every vertex just once, and a set $\mathscr{R}$ of (not necessarily disjoint) 4-cycles containing every vertex. Assume that when two of the 4-cycles in $\mathscr{R}$ intersect, the intersection is a single vertex; and when one of the cycles in $\mathscr{C}$ intersects a 4-cycle in $\mathscr{R}$, the intersection is a common edge. Also assume every edge of G lies on either one of the cycles in $\mathscr{C}$ or on one of the 4-cycles in $\mathscr{R}$ (or both). Then there is a two-way infinite Hamiltonian path in G.*

Proof. For convenience, a cycle in $\mathscr{C}$ is called a *polygon*, and a 4-cycle in $\mathscr{R}$ is called a *rectangle*. The basic idea is to 'annex' polygons one at a time to construct a sequence $C_1, C_2, \ldots$ of longer and longer cycles whose limit is a two-way infinite Hamiltonian path in G. We say that two subsets of G are *adjacent* if they are joined by an edge.

Step 1. *The polygons can be arranged in a list $\mathscr{P}_1, \mathscr{P}_2, \ldots$ so that, for some strictly increasing sequence of natural numbers $1 = a_0 < a_1 < \cdots$, we have*:

(1) *for every $n \geqslant 0$, the subgraph $X_n = G\backslash(\mathscr{P}_1 \cup \cdots \cup \mathscr{P}_{a_n - 1})$ is connected; and*

(2) *for every $n \geqslant 0$, if $i > a_n$, then $\mathscr{P}_i$ is adjacent to one of the polygons $\mathscr{P}_{a_n}, \mathscr{P}_{a_n+1}, \ldots, \mathscr{P}_{i-1}$.*

Let $\mathscr{P}_1$ be any polygon and construct the rest of the list by induction. The subgraph X_n, like G, has only one end, so there is one infinite component in $X_n\backslash\mathscr{P}_{a_n}$ and (at most) finitely many finite components. Since X_n is connected, we may list the polygons $\mathscr{P}_{a_n+1}, \mathscr{P}_{a_n+2}, \ldots, \mathscr{P}_{a_{n+1}-1}$ in the finite components, in such a way that each $\mathscr{P}_i$ is adjacent to one of the preceding polygons in X_n. Because X_n is connected, there is a polygon $\mathscr{P}_{a_{n+1}}$ in X_{n+1} adjacent to $X_n\backslash X_{n+1}$. A bit of care in selecting $\mathscr{P}_{a_{n+1}}$ at each stage assures that every polygon eventually occurs in the list.

Step 2: *There is a sequence $C_1, C_2, \ldots$* of cycles of G and a sequence $e_1, e_2, \ldots$ *of edges of G such that*:

(1) *the vertex set of C_i is $\mathscr{P}_1 \cup \cdots \cup \mathscr{P}_i$*;

(2) *e_{i+1} is an edge of C_i for each i*;

(3) *the path $C_i\backslash e_{i+1}$ is contained in C_{i+1}; and*

(4) *if $i > a_n$, then $e_i \in X_n$.*

Given C_i, we construct C_{i+1} and e_{i+1} inductively. Let n be maximal with $i \geqslant a_n$. Then $i + 1 > a_n$, so $\mathscr{P}_{i+1}$ is adjacent to some polygon $\mathscr{P}_j$ with $a_n \leqslant j \leqslant i$. The edge joining $\mathscr{P}_{i+1}$ to $\mathscr{P}_j$ does not lie on any polygon, so it must lie on some rectangle R_i. Then R_i has an edge e_{i+1} in common with C_i, and an edge in common with $\mathscr{P}_{i+1}$. The symmetric difference $C_{n+1} = C_n \triangle R_n \triangle \mathscr{P}_{n+1}$ is the desired cycle.

Step 3: *Let P_n be the path $C_{a_n-1} \backslash e_{a_n}$. Then the union P_∞ of the paths $P_1, P_2, \ldots$ is a two-way infinite Hamiltonian path in G.* By conditions 3 and 4 of Step 2, we see that $P_1 \subset P_2 \cdots$, so P_∞ is a (one-way infinite or two-way infinite) path in G. It follows from condition 4 of Step 2 that P_∞ is not one-way infinite. Because condition 1 of Step 2 implies P_∞ covers every vertex of G, we conclude that it is a two-way infinite Hamiltonian path. □

Corollary 3.2. *There is a two-way infinite directed Hamiltonian path in* $\mathrm{Cay}([p,q]; T_p, T_q, T_2)$.

Proof. Since each generator has order 2, the Cayley digraph can be viewed as an undirected graph. The relation $(T_q, T_2)^p = e$ gives a collection of $2p$-gons, and the relation $(T_pT_q)^2 = e$ gives rectangles. □

Corollary 3.3 (of proof). *There is a two-way infinite directed Hamiltonian path in* $\mathrm{Cay}([p,q]^+; R, S^{-1})$.

Proof. The relation $R^p = e$ gives oriented p-gons in the Cayley digraph. The relation $(RS)^2 = e$ gives rectangles whose arcs are oriented in such a way that they can be used to annex the p-gons into longer and longer circuits as in the proof of the theorem (see Fig. 1). □

4. One-way infinite directed Hamiltonian paths in $\mathrm{Cay}(R, S)$ and $\mathrm{Cay}(R, S, T)$

Theorem 4.1. $\mathrm{Cay}([p,q]^+; R, S^{-1})$ *has a one-way infinite directed Hamiltonian path if* $p, q > 3$.

This entire section (except Corollary 4.7) is devoted to the proof of Theorem 4.1. From $\mathrm{Cay}(R, S)$, it is easy to construct a certain filtration of the hyperbolic plane; the

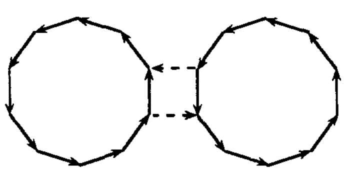


Fig. 1.

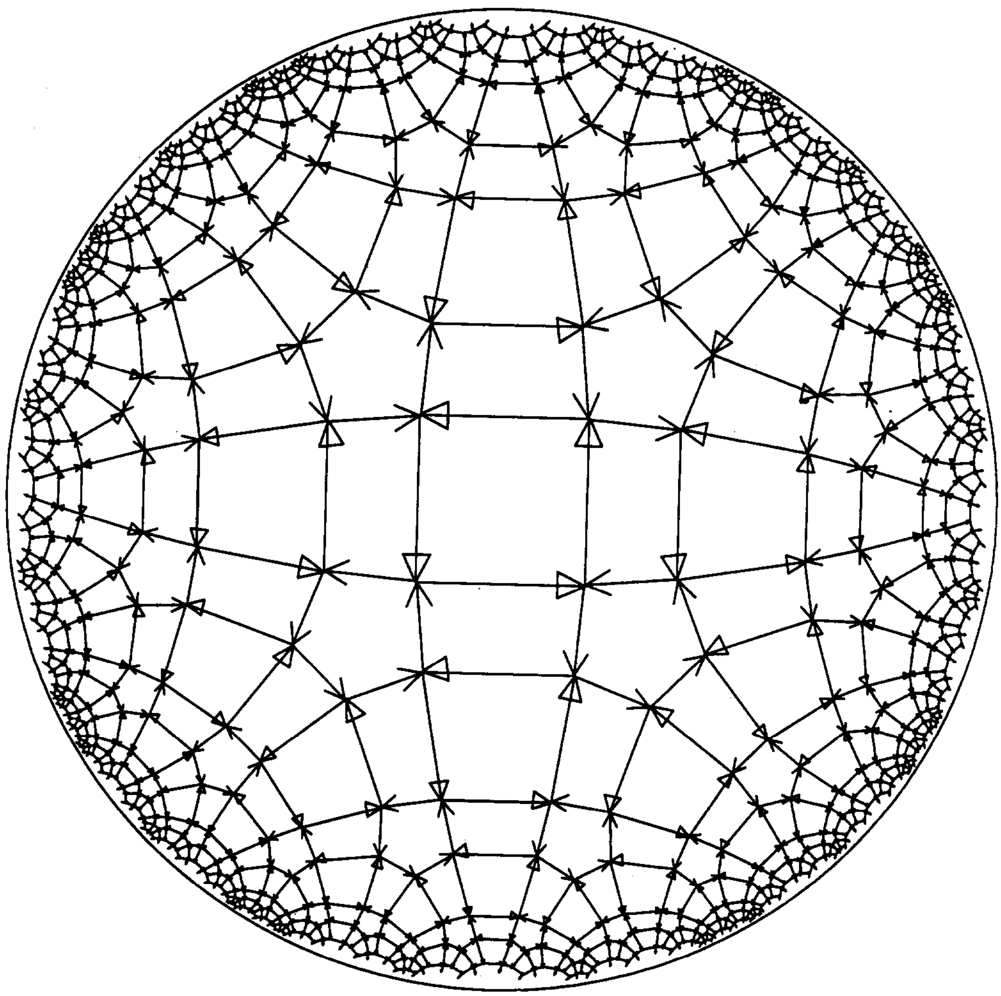


Fig. 2. Cay($[p,q]^+$; R,S).

work (rather pedestrian) comes in showing that the boundary of each filtration set is a circuit in Cay(R,S), and that, as pictured in Fig. 6, these circuits can be joined to form a one-way infinite directed Hamiltonian path. Whenever we use a similar argument in later sections, we will leave more of the work to the reader.

Assumption. By interchanging p and q if necessary, we will assume $p \geqslant 5$ and $q \geqslant 4$.

Remark. Cay(R,S) tessellates the hyperbolic plane into p-gons, q-gons, and squares, coming from the relations $R^p = e$, $S^q = e$, and $RSRS = e$, respectively (see Fig. 2). (If $q = 4$, we can distinguish q-gons from squares by noting that q-gons are the quadrilaterals oriented in the same direction (say, counterclockwise) as p-gons.) Note certain properties of this tessellation:

(1) each vertex has indegree 2 and outdegree 2, and inarcs and outarcs alternate in cyclic order around the vertex (i.e., in cyclic order around any vertex starting, say, at an inarc, one encounters an inarc, an outarc, the other inarc and the other outarc, in that order).
(2) each vertex is adjacent to a p-gon, a q-gon, and two squares; and
(3) each arc has to its right a square, and to its left a p-gon or q-gon.

Definition. Two polygons of a tessellation are *edge-adjacent* if they have an edge in common.

Definition. Use the Cay(R, S)-tessellation to define a filtration of the hyperbolic plane by letting A_1 be the union of a single p-gon and all the squares edge-adjacent to it, and letting A_n be the union of A_{n-1}, all p-gons and q-gons edge-adjacent to A_{n-1}, and all squares edge-adjacent to these p-gons and q-gons. Note that this is not quite the filtration described in Definition 2.1.

Definition. The boundary of A_n is denoted by ∂_n.

We will see that ∂_n is connected (i.e., A_n is simply connected). In fact, ∂_n is a circuit in Cay(R, S). These and other facts are rendered more obvious by consideration of the following auxiliary graph.

Definition. Construct a planar graph — the *square-dual* of Cay(R, S) — by placing a vertex at the center of each square of Cay(R, S), and connecting two of these vertices by an edge (drawn as a hyperbolic line segment) if and only if the corresponding squares share a vertex in Cay(R, S). As pictured in Fig. 3, the square-dual tessellates the hyperbolic plane into p-gons and q-gons, with two p-gons and two q-gons meeting at each vertex. Thus the square-dual is the quasi-regular tessellation $\{{p \atop q}\}$.

Definition. In accordance with Definition 2.1, the square-dual determines a filtration of the hyperbolic plane in which the first filtration set is a single p-gon. The nth filtration set is denoted $A_n^{\square}$, to avoid confusion with the Cay(R, S)-filtration.

Lemma 4.2. *$A_n^{\square}$ is convex.*

Proof. $A_1^{\square}$ is a regular p-gon, so it is (hyperbolically) convex. The boundary of $A_n^{\square}$ is a polygon, so if $A_n^{\square}$ is not convex, then, at some vertex of the boundary, the interior angle is greater than π. In this case, the boundary of $A_n^{\square}$ must contain two consecutive sides $\overline{ab}$ and $\overline{bc}$ of some p-gon or q-gon P not contained in $A_n^{\square}$. Assume, for simplicity, that P is a p-gon (see Fig. 4). Let Q_a and Q_c be the q-gons containing $\overline{ab}$ and $\overline{bc}$, respectively. Since Q_a is in $A_n^{\square}$, it shares at least a vertex with $A_{n-1}^{\square}$.

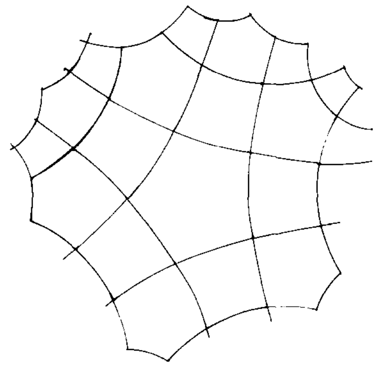


Fig. 3. Square-dual of Cay(R, S) $(p = 4, q = 5)$.

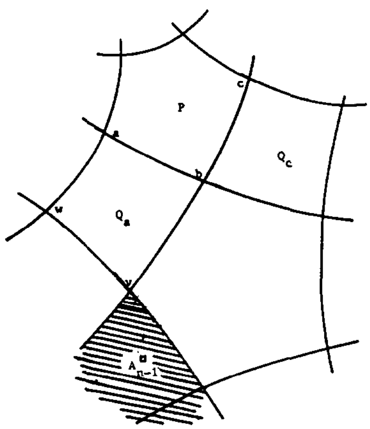


Fig. 4.

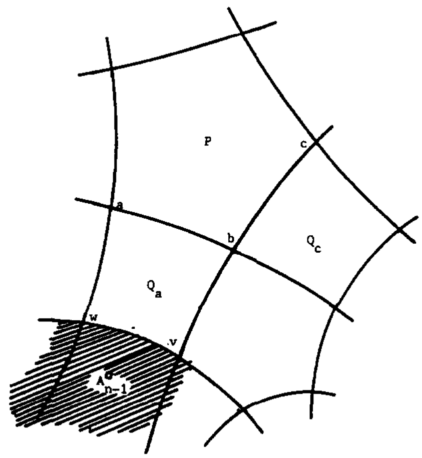


Fig. 5.

Suppose first that Q_a shares an edge $\overline{vw}$ with $A_{n-1}^{\square}$. By induction, $A_{n-1}^{\square}$ is convex, so all of it lies on the opposite side of $\overleftrightarrow{vw}$ from Q_a. However, it is clear that the hyperbolic line $\overleftrightarrow{ab}$ never intersects $\overleftrightarrow{vw}$, and the region between these lines separates Q_c from $A_{n-1}^{\square}$. This implies that Q_c is not in $A_n^{\square}$, which is a contradiction. If Q_a shares only a vertex v with $A_{n-1}^{\square}$, it is still possible to choose an edge $\overleftrightarrow{vw}$ of Q_a whose extension $\overleftrightarrow{vw}$ separates Q from $A_{n-1}^{\square}$ and does not intersect $\overleftrightarrow{ab}$ (see Fig. 5), so the same argument applies. $\square$

Remark. Lemma 4.2 implies $A_n^{\square}$ is simply connected. Because a square (resp., a *p-gon* or q-gon) is contained in A_n iff the corresponding vertex (resp., p-gon or q-gon) is in $A_n^{\square}$, it follows that A_n is simply connected. This shows ∂_n is a simple closed curve, so ∂_n is a cycle in the graph underlying Cay(R, S).

Definition. The boundary ∂_n divides the hyperbolic plane into two regions. The finite region, say A_n, is the *inside* of ∂_n; the other region is the *outside* of ∂_n.

Lemma 4.3. *The boundary ∂_n is edge-adjacent only to squares on the inside, and only to p-gons and q-gons on the outside.*

Proof. For any p-gon (or q-gon) P contained in A_n, it follows from the construction of A_n that every square edge-adjacent to P is also in A_n. Hence, each edge of P is in the interior of A_n and, therefore, does not lie on the boundary ∂_n. This implies the first half of the lemma, namely, that ∂_n is edge-adjacent only to squares on the inside. Because every arc of the digraph is an edge of one square and an edge of one p-gon or q-gon, the second half of the lemma follows from the first. $\square$

Lemma 4.4. *The boundary ∂_n is a circuit in the digraph* Cay(R, S). *Its orientation is clockwise.*

Proof. To the right of each arc of ∂_n is a square, and Lemma 4.3 shows this square is on the inside of ∂_n. Therefore, arc of ∂_n is oriented so that the inside is to the right, so ∂_n is a clockwise circuit. $\square$

Lemma 4.5. *Every vertex of* Cay(R, S) *lies on the boundary of some filtration set A_n.*

Proof. Any vertex v is in A_n, for some n. Assume $v \notin A_{n-1}$ and suppose, for a contradiction, that $v \notin \partial_n$. The p-gon, the q-gon, and the two squares adjacent to v are in A_n, but none are in A_{n-1}. Corresponding to the p-gon, the q-gon, and the two squares, the square-dual has a p-gon P, a q-gon Q, and two vertices a and b; these are in $A_n^{\square}$, but not in $A_{n-1}^{\square}$. (The edge $\overline{ab}$ is disjoint from $A_{n-1}^{\square}$.) Now, some vertex x of P must be in $A_{n-1}^{\square}$, as must be some vertex y of Q. (The line segment $\overline{xy}$ is contained in $A_{n-1}^{\square}$.) Since $P \cup Q$ is convex and $\overline{ab}$ is the only edge that P and Q share, the hyperbolic line segment $\overline{xy}$ must intersect edge $\overline{ab}$. But $\overline{xy}$ is contained in $A_{n-1}^{\square}$ and $\overline{ab}$ is disjoint from $A_{n-1}^{\square}$. This is a contradiction. $\square$

Remark 4.6. (1) As the circuit ∂_n passes through a vertex v, it must make either a 'left turn' or a 'right turn'. (It cannot go 'straight ahead' because that is another inarc of v.)

(2) Two consecutive vertices of ∂_n cannot both be left turns. For, if both $\overrightarrow{abc}$ and $\overrightarrow{bcd}$ are left turns, then the three vertices of the square-dual corresponding to the squares on $\overrightarrow{ab}$, $\overrightarrow{bc}$, and $\overrightarrow{cd}$ form a concave angle, contradicting the convexity of the nth filtration set in the square-dual.

(3) Suppose a and b are consecutive vertices of ∂_n, with a right turn at a and a left turn at b, and let b' be the vertex to which a left turn at a would have brought us. Then $b' \in \partial_{n+1}$. (Because $A_n^{\square}$ is convex, $b' \notin A_n$.)

(4) If there is an arc $\overrightarrow{ab}$, with $a \in \partial_n$ and $b \in \partial_{n+1}$, then ∂_{n+1} must make a left turn at b.

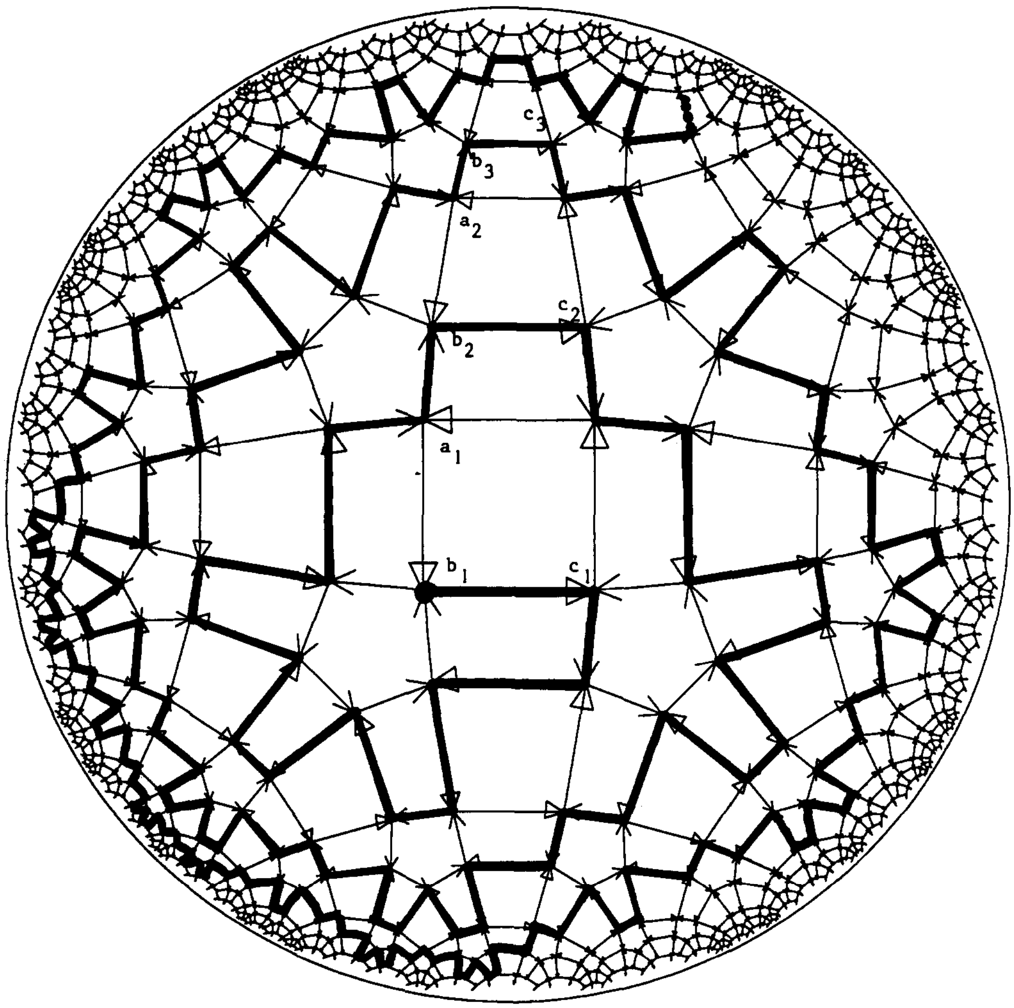


Fig. 6. A one-way infinite directed Hamiltonian path in Cay(R, S).

Proof of Theorem 4.1. Let $\overrightarrow{a_1 b_1 c_1}$ be a left turn of ∂_1. Then Remark 4.6(2) implies a_1 must be a right turn of ∂_1, so, by Remark 4.6(3), there is an arc $\overrightarrow{a_1 b_2}$ with $b_2 \in \partial_2$. Remove arc $\overrightarrow{a_1 b_1}$ from ∂_1 and, in its stead, insert arc $\overrightarrow{a_1 b_2}$. Now Remark 4.6(4) implies b_2 is a left turn $(\overrightarrow{a_2 b_2 c_2})$ of ∂_2. Replace $\overrightarrow{a_2 b_2}$ with $\overrightarrow{a_2 b_3}$, where $b_3 \in \partial_3$. Continue, removing an arc $\overrightarrow{a_n b_n}$ from each ∂_n in turn, and inserting an arc $\overrightarrow{a_n b_{n+1}}$, with $b_{n+1} \in \partial_{n+1}$. This produces a one-way infinite directed Hamiltonian path in Cay(R, S) (see Fig. 6). $\square$

Corollary 4.7. *There is a one-way infinite directed Hamiltonian path in* Cay$([p, q]^+; R, S, T)$.

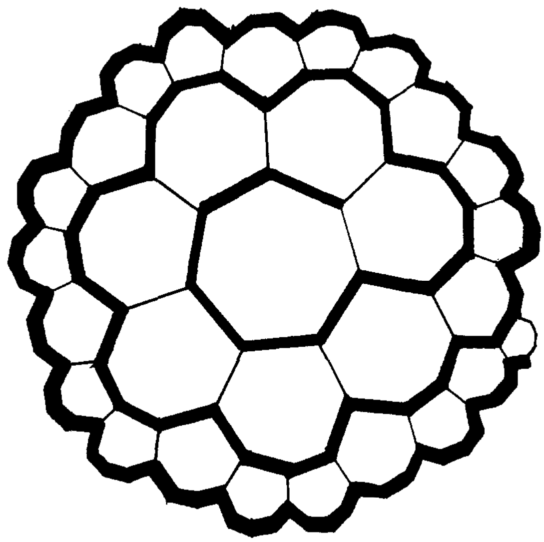


Fig. 7. A one-way infinite Hamiltonian path in $(q, 3)$.

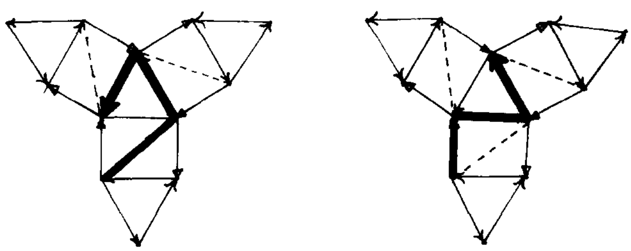


Fig. 8. (a) A left turn at P, (b) a right turn at P.

Proof. If $p, q > 3$, then the theorem asserts there is a one-way infinite directed Hamiltonian path in Cay(R, S), and hence in Cay(R, S, T). So, by interchanging p and q if necessary, we may assume $p = 3$. In this case, contracting each p-gon of Cay$([3, q]^+; R, S, T)$ to a point yields the regular tessellation $\{q, 3\}$ and, in accordance with Definition 2.1, this tessellation determines a filtration of the hyperbolic plane in which A_1 is a single q-gon. Fig. 7 shows a one-way infinite Hamiltonian path in $\{q, 3\}$ formed by joining the boundaries of the filtration sets. A one-way infinite directed Hamiltonian path in Cay$([3, q]^+; R, S, T)$ can be constructed by visiting the p-gons of Cay(R, S, T) in the order indicated by the path in $\{q, 3\}$. (At each p-gon, the path in the tessellation makes either a left turn or a right turn. Fig. 8 shows, in each case, which arc should be used to enter P, and how P should be traversed.) For example, Fig. 9 depicts the resulting directed Hamiltonian path in Cay$([3, 7]^+; R, S, T)$. □

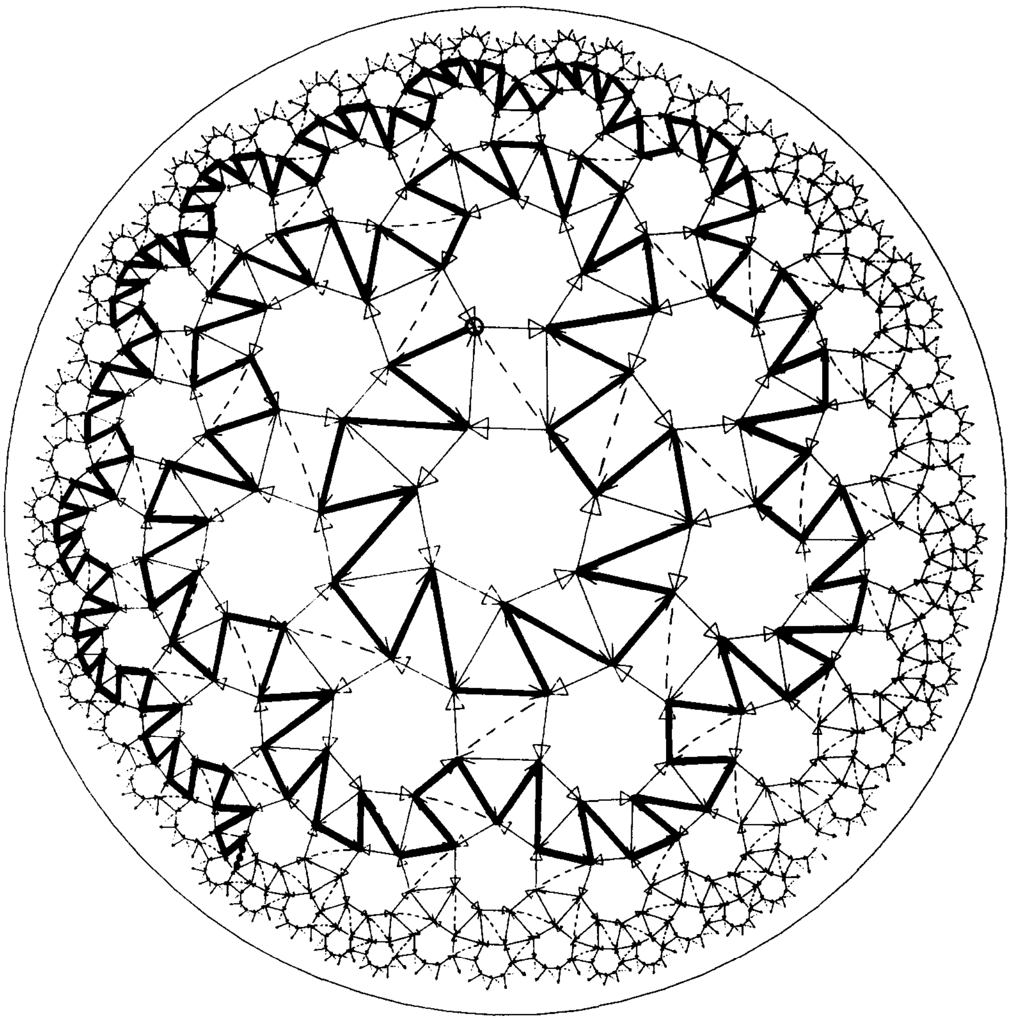


Fig. 9. A one-way infinite Hamiltonian path in Cay($[3,7]^+$; R,S,T).

5. A one-way infinite directed Hamiltonian path in Cay($[p,q]$)

Definition. Let $\{p,q,2\}$ be the regular tessellation of the hyperbolic plane into congruent right triangles with angles π/p, π/q, $\pi/2$. Note that $\{p,q,2\}$ is a common refinement of the dual tessellations $\{p,q\}$ and $\{q,p\}$.

Definition. In accordance with Definition 2.1, the tessellations $\{p,q\}$, $\{q,p\}$ and $\{p,q,2\}$ each determine a filtration of the hyperbolic plane. To avoid confusion, the nth filtration set is denoted $A_n^{\{p,q\}}$, $A_n^{\{q,p\}}$, or $A_n^{\{p,q,2\}}$, as appropriate. We specify the first filtration set as follows: $A^{\{p,q\}}$ is a single p-gon, $A^{\{q,p\}}$ is the union of the p q-gons sharing some fixed vertex, and $A_1^{\{p,q,2\}}$ is the regular p-gon formed by the union of the $2p$ triangles sharing a given vertex.

Lemma 5.1. $A_{2k-1}^{\{p,q,2\}} = A_k^{\{p,q\}}$ *and* $A_{2k}^{\{p,q,2\}} = A_k^{\{q,p\}}$.

Lemma 5.2. *Each filtration set* $A_n^{\{p,q\}}$, $A_n^{\{q,p\}}$, *or* $A_n^{\{p,q,2\}}$ *is simply connected (in fact, if* $p,q > 3$, *they are convex). Furthermore, every vertex of* $\{p,q\}$, $\{q,p\}$, *or* $\{p,q,2\}$ *lies on the boundary of some filtration set* $A_n^{\{p,q\}}$, $A_n^{\{q,p\}}$, *or* $A_n^{\{p,q,2\}}$, *respectively.*

Proof. For the $\{p,q\}$-filtration and the $\{q,p\}$-filtration, these results are probably widely known. (They can be verified inductively by a careful study of the transition from one filtration set to the next — in the spirit of the proof of Proposition 6.1.) Using Lemma 5.1, it is then easy to deduce the results for the $\{p,q,2\}$-filtration. □

Lemma 5.3. *Let* $\triangle PQT$ *be a triangle of the tessellation* $\{p,q,2\}$, *and assume* $\triangle PQT \subset A_n^{\{p,q,2\}}$ *but* $\triangle PQT \not\subset A_{n-1}^{\{p,q,2\}}$. *Then some vertex of* $\triangle PQT$ *is on the boundary of* $A_n^{\{p,q,2\}}$.

Proof. Assume the angles of $\triangle PQT$ at P, Q, T are π/p, π/q, and $\pi/2$, respectively, and assume, for simplicity, that $n = 2k$ is even. Because $\triangle PQT \not\subset A_{2k-1}^{\{p,q,2\}}$, we know P is not on the boundary of $A_{2k-2}^{\{p,q,2\}} = A_{k-1}^{\{q,p\}}$. Because $P \in A_{2k}^{\{p,q,2\}} = A_k^{\{q,p\}}$, and every vertex is on the boundary of some filtration set of the $\{q,p\}$-filtration, we conclude that P is on the boundary of $A_k^{\{q,p\}} = A_{2k}^{\{p,q,2\}}$. □

Theorem 5.4. *There is a one-way infinite directed Hamiltonian path in* $\mathrm{Cay}([p,q]; T_p, T_q, T_2)$.

Proof. Since each generator has order 2, the Cayley digraph can be viewed as an undirected graph. The graph tessellates the hyperbolic plane into $2p$-gons, $2q$-gons, and squares. In accordance with Definition 2.1, this determines a filtration in which the first filtration set is a single $2p$-gon. Note that $\mathrm{Cay}([p,q])$ is the planar dual of the tessellation $\{p,q,2\}$. One can check that a vertex (resp. polygon) of $\mathrm{Cay}([p,q])$ belongs to the nth filtration set of the $\mathrm{Cay}([p,q])$-filtration iff the corresponding triangle (resp. vertex) of $\{p,q,2\}$ is in the nth filtration set (resp. the $(n-1)$st filtration set) of the $\{p,q,2\}$-filtration. Hence, each filtration set of the $\mathrm{Cay}([p,q])$-filtration is simply connected and every vertex is on the boundary of some filtration set. If $p,q \geqslant 4$, it is easy (see Fig. 10) to join these boundaries to form a one-way infinite Hamiltonian path. On the other hand, if $p = 3$ (or, similarly, if $q = 3$), a naive attempt to join the boundaries into a one-way infinite Hamiltonian path will fail, but Fig. 11 depicts a solution to this problem: as the path enters the boundary of any even-numbered filtration set, it must begin by detouring around three sides of a square; the path then continues normally around the boundary of the filtration set; on the boundary of the

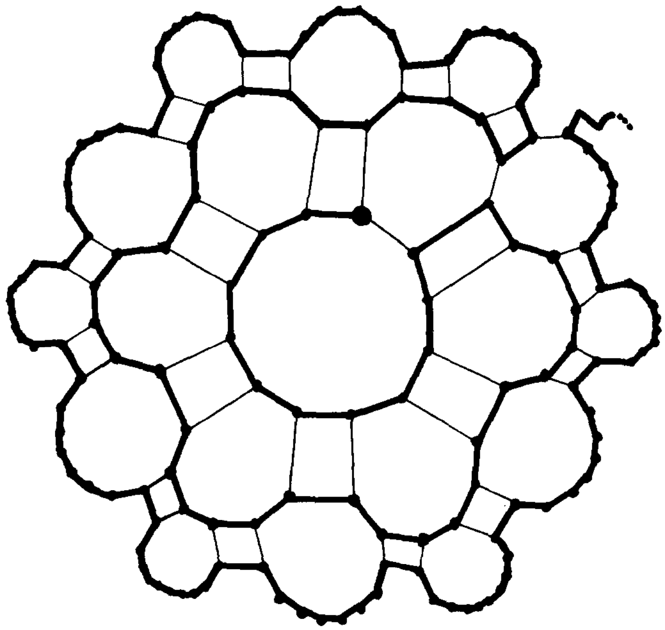


Fig. 10. A one-way infinite Hamiltonian path in Cay([p,q]) for $p,q \geqslant 4$.

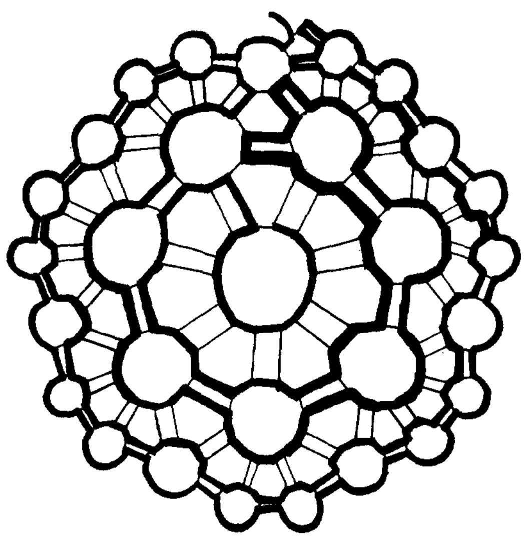


Fig. 11. A one-way infinite Hamiltonian path in Cay([p,q]) for $q = 3$ (the first two detours appear in this figure).

next filtration set (odd-numbered), the path begins normally but exits slightly early because the first two vertices of this boundary were covered by the detour in the previous stage. □

6. Two-way infinite directed Hamiltonian paths in Cay(R, T) and Cay(R, S, T)

Definition. A subset A of the hyperbolic plane is *unbounded* if the closure of A is not compact.

Definition. A closed, unbounded subset X of the hyperbolic plane has *only one end* if, for every compact subset K of X, the complement $X \backslash K$ of K in X has only one unbounded component.

Proposition 6.1. *If $p \geqslant 4$, then there is an infinite collection $\{Q_1, Q_2, \dots\}$ of (closed) q-gons of the tessellation $\{q, p\}$ such that:*

(1) *the union of the q-gons in the collection contains every vertex of $\{q, p\}$;*

(2) *the union has only one end and is connected and simply connected; and*

(3) *no two of the q-gons share an edge.*

Proof. It suffices to find a sequence $\{Q_1, Q_2, \dots\}$ of q-gons such that:

(1) Q_{n+1} intersects Q_n in a single vertex, and is disjoint from $Q_1, Q_2, \dots, Q_{n-1}$; and

(2) every vertex of $\{q, p\}$ is in some q-gon Q_n.

Such a sequence can be constructed by induction. In each q-gon Q_i, we will designate a vertex to be the 'special vertex', and Q_{i+1} will intersect Q_i in this special vertex. Begin by letting Q_i be any q-gon, and designating one of its vertices to be the first special vertex. Inductively, let X_n be the union of all the q-gons that intersect any of $Q_1, \dots, Q_n$ anywhere other than at the nth special vertex. (These q-gons may intersect the nth special vertex as well.) (See Fig. 12.) Assume inductively that the nth special vertex is on the boundary of X_n, so there are some q-gons that contain the special vertex but are not contained in X_n. Let Q_{n+1} be the rightmost (i.e., the most clockwise) of these q-gons. Let X_n^* be the union of all the q-gons (including Q_{n+1}) that intersect any of $Q_1, \dots, Q_n$ or that share an edge with Q_{n+1}, and let the new (i.e., the $(n+1)$st special vertex be the last (i.e., the most clockwise) vertex of Q_{n+1} on the boundary of X_n^*. We must prove that, at each stage, such a special vertex exists (i.e., that there is some vertex of Q_{n+1} on the boundary of X_n^*), and that every vertex belongs to some q-gon in the sequence.

Let ∂_n be the boundary of X_n. Each vertex in ∂_n lies on some number of q-gons in X_n, and we can arrange these numbers in a (circular) sequence by proceeding clockwise around ∂_n. This sequence is called σ_n.

Case 1: $q = 3$. We will show by induction on n that (a) every term of σ_n is either 2 or 3, except that the term for the special vertex is 4 or 5; (b) the term of σ_n immediately preceding the term for the special vertex is 2; (c) a new special vertex can be

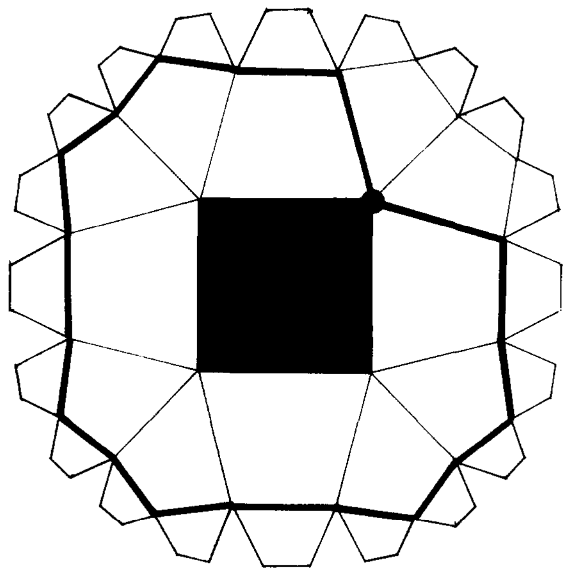


Fig. 12. Q_1 is shaded; X_1 is outlined; the first special vertex is a black dot.

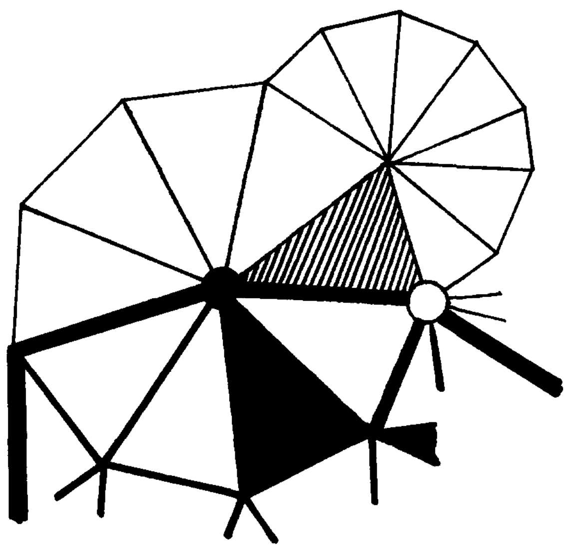


Fig. 13. The transition in Case 1 ($q = 3$). Q_n is shaded; ∂_n is dark; the nth special vertex is a black dot; the $(n+1)$st special vertex is a white dot. $\sigma_n = \ldots 2\ k\ m \ldots (m = 2 \text{ or } 3) \Rightarrow \sigma_{n+1} = \ldots 322 \ldots 232 \ldots 2\ m+2 \ldots$

selected; and (d) every vertex in the interior of X_n is on some q-gon Q_i. (Notice that, because every vertex is in the interior of some X_n, (d) implies every vertex belongs to some q-gon in the collection.) Fig. 13 depicts the transition from ∂_n to ∂_{n+1} in a neighborhood of the nth special vertex (the rest of the boundary does not change).

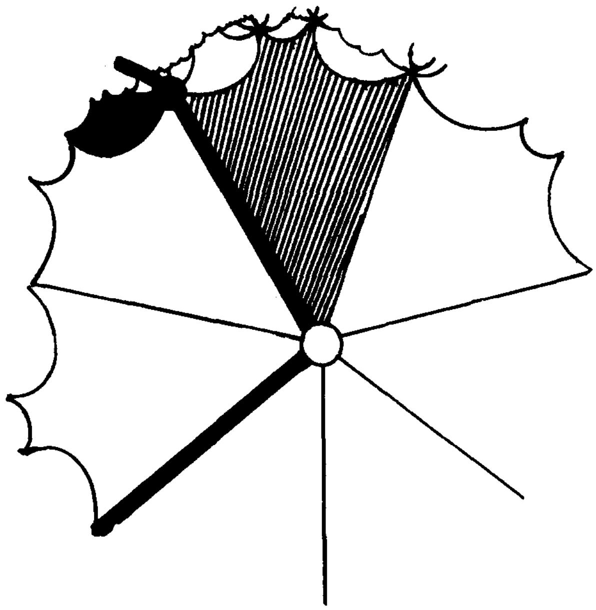


Fig. 14. The transition in Case 2 ($p \geqslant 6$ and $q \geqslant 4$). $\sigma_n = \ldots 1$ k $m \ldots \Rightarrow \sigma_{n+1} = \ldots 2$ [1's and 2's] 1 $m+2$ $\ldots$

The vertex preceding the nth special vertex on ∂_n receives a third q-gon. The nth special vertex is in the interior of X_{n+1}, so does not contribute a term to σ_{n+1}. The vertex following the nth special vertex on ∂_n is the new special vertex, and receives two additional q-gons (Q_{n+1} and a q-gon containing an edge of Q_{n+1}). A number of new vertices appear on ∂_{n+1}. By inspection, each of these new vertices is on two q-gons of X_{n+1}, except for one on three q-gons. (The vertex immediately preceding the new special vertex of ∂_{n+1} is on two q-gons.) It is also easy to see from the figure that any vertex in the interior of X_{n+1}, but not in the interior of X_n, lies on Q_{n+1}.

Case 2: $p \geqslant 6$ *and* $q \geqslant 4$. One can show by an argument similar to the above (see Fig. 14) that every term of σ_n is 1 or 2, except that the term for the special vertex is 3 or 4. (Also every interior vertex of X_n is on a chosen q-gon.) This is sufficient.

Case 3: $p = 5$ *and* $q \geqslant 5$. Much as before, one can show every term of σ_n is 1 or 2, except that the term for the special vertex is 3 or 4. But there is something a bit new. Namely, when the term for the nth special vertex of σ_n is 4, the situation is not as depicted in Fig. 14, because Q_{n+1} contains both of the vertices next to the nth special vertex on ∂_n (see Fig. 15). So one needs to note (inductively) that the term for the special vertex is immediately preceded by two consecutive 1's in σ_n.

Case 4: $p = 4$ (*and hence* $q \geqslant 5$). Fig. 16 illustrates the possible transitions from X_n to X_{n+1}. (This case and the following one are somewhat more complicated than the preceding ones.) To see that the transitions considered there are exhaustive, note inductively that: (a) the term for the special vertex is 3; (b) the term for the special

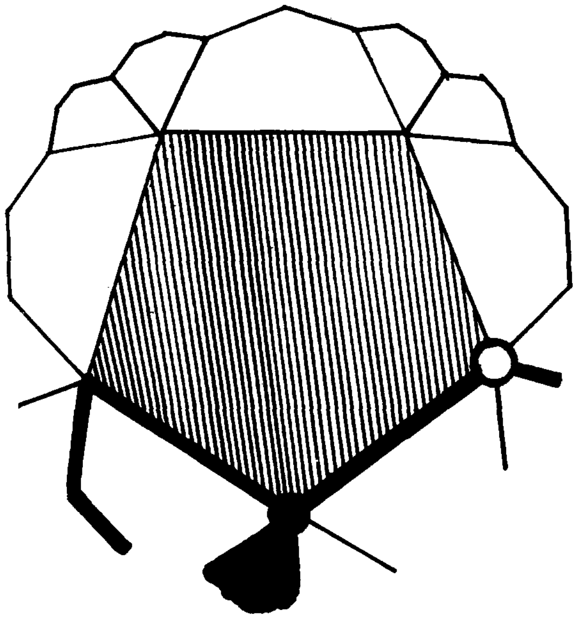


Fig. 15. The transition in Case 3 ($p = 5$ and $q \geqslant 5$). $\sigma_n = \ldots 1\ 1\ k\ m \ldots \Rightarrow \sigma_{n+1} = \ldots 2$ [1's and 2's] $1\ 1\ m+2 \ldots$

vertex is immediately preceded by two consecutive 1's in σ_n; (c) except the term for the special vertex and possibly one other 3 somewhere, every term of σ_n is 1 or 2; and (d) if 3 occurs other than for the special vertex, the only 1's occur between the term for the special vertex and this other 3.

Case 5: $p = 5$ *and* $q = 4$. The possible transitions are illustrated in Fig. 17. Note that: (a) every term of σ_n is 1, 2 or 3 except that the term for the special vertex is either 3 or 4; (b) the four terms immediately preceding the term for the special vertex are 2, 1, 2, 1; (c) if the term for the special vertex is 4, then it is immediately followed by either 1 or 2; and (d) a 3 occurring anywhere other than for the special vertex is immediately preceded either by the term for the special vertex (which must be 3) or by 1, and must be immediately followed either by 1, 2 or 2, 1, 2.

In Cases 1–4, X_n was convex except perhaps for a concave angle at the special vertex and perhaps one other vertex. It is therefore easy to see that, in expanding from X_n to X_{n+1}, the additional q-gons do not collide with some other part of the boundary. In Case 5 there may be a number of concave angles — one for each 3 in σ_n — but it is again easy to rule out the possibility of collision because each 3 is surrounded by a 1 on each side. Adjoining two extra q-gons at each 3-vertex of ∂_n (see Fig. 18) expands X_n to a set $\tilde{X}_n$ that is convex except perhaps for a concave angle at the special vertex. □

Theorem 6.2. *There is a two-way infinite directed Hamiltonian path in* $\mathrm{Cay}([p,q]^+; R, T)$ *if* $p \geqslant 4$.

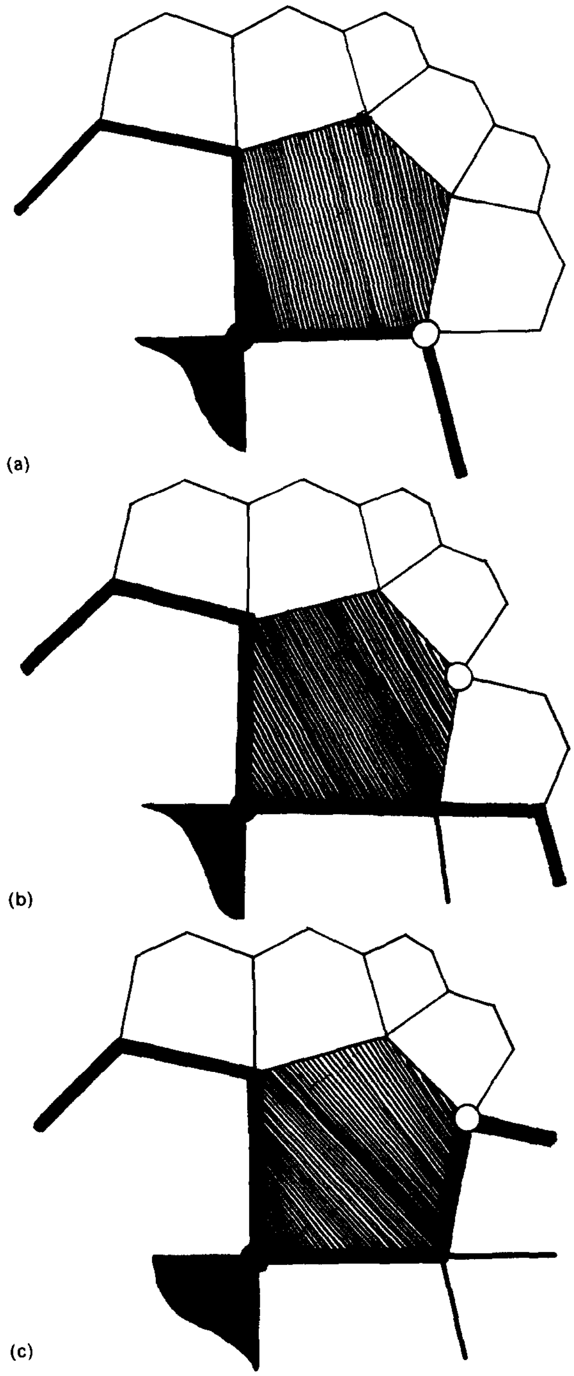
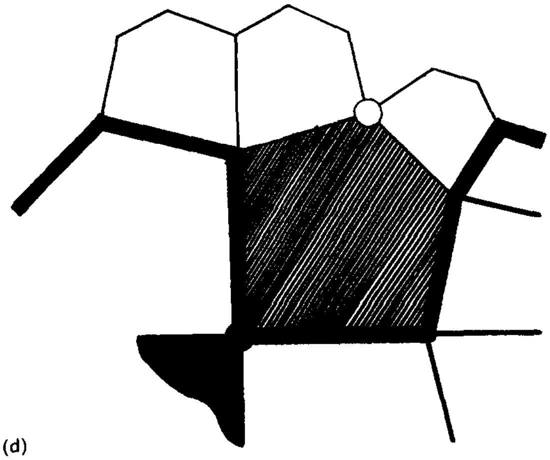
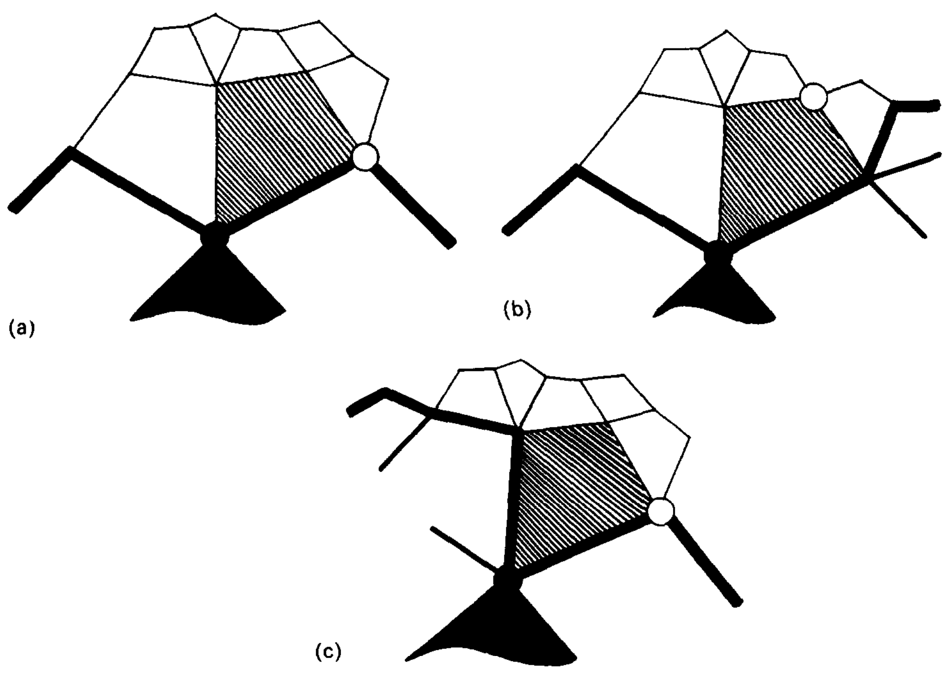


Fig. 16. The transition in Case 4 ($p = 4$ and $q \geqslant 5$). (a) $\sigma_n = \ldots 1\ 1 3\ 1 \Rightarrow \sigma_{n+1} = 2$ [1's and 2's] 1 1 3. (b) $\sigma_n = \ldots 1\ 1 3\ 2\ m$ ($m = 1$ or 2) $\Rightarrow \sigma_{n+1} =$ [1's and 2's] 1 1 3 1 1…1 $m + 1$. (c) $\sigma_n = \ldots 1\ 1\ 3\ 3\ 1 \ldots \Rightarrow \sigma_{n+1} =$ [1's and 2's] 1 1 3… (d) $\sigma_n = \ldots 1\ 1\ 3\ 3\ 2\ m \ldots$ ($m = 1$ or 2) $\Rightarrow \sigma_{n+1} = \ldots 2\ 1 \ldots 1\ 2\ 1 \ldots 1\ 3\ 1 \ldots 1\ m + 1 \ldots$



(d)

Fig. 16d.



(a) (b) (c)

Fig. 17. The transition in Case 5 ($p = 5$ and $q = 5$). (a) To get 4 at the new special vertex, must have $m = 2$, so next is *not* 3. $\sigma_n = \ldots 1$ ③ $m \ldots$ ($m = 1$ or 2) $\Rightarrow \sigma_{n+1} = \ldots 2\ 2$ [1's and 2's] 2121 $m + 2$ (b) $\sigma_n = \ldots 1$ ③ $3\ m \Rightarrow \sigma_{n+1} = \ldots 2\ 2\ 1\ 2\ 1\ 2\ 1$ ③ $1\ m + 1$. So ③ 3 2 1 2 transforms to ③ 1 3 1 2 and ③ 3 1 2 transforms to ③ 1 2 2. (c) $\sigma_n = \ldots 1\ 2\ 1$ ④ $m \ldots$ ($m = 1$ or 2) $\Rightarrow \sigma_{n+1} \ldots 1\ 3\ 1\ 2$ [1's and 2's] 2 1 2 1 $m + 2$. To get 4 at the new special vertex, must have $m = 2$, so the next vertex is *not* 3.

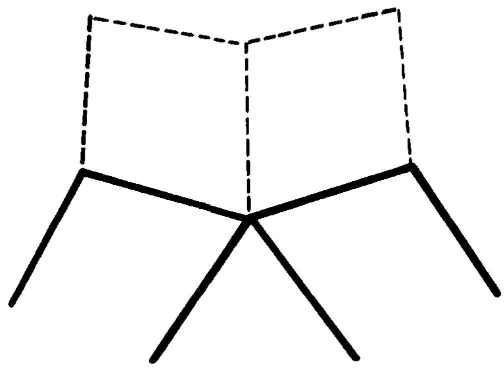


Fig. 18. X_n is solid; $\tilde{X}_n$ is dotted.

Proof. Cay(R, T) tessellates the hyperbolic plane into p-gons (oriented counterclockwise, say) and $2q$-gons (see Fig. 30). Contracting each p-gon of the tessellation to a point yields the regular tessellation $\{q, p\}$. Let I be the (closed) union of all the p-gons of the Cay(R, T)-tessellation, together with the $2q$-gons corresponding to the q-gons belonging to the collection described in Proposition 6.1. It is immediate that:

(1) I contains every p-gon;

(2) I has only one end and is connected and simply connected; and

(3) I does not contain any two $2q$-gons that share an edge.

We claim the boundary of I is the desired directed Hamiltonian path in Cay(R, T). Condition (1) implies every vertex is in I, while Condition (3) shows no vertex is in the interior of I. Therefore, every vertex is on the boundary of I. Condition (2) implies that the boundary is connected. Thus the boundary is a (two-way infinite) Hamiltonian path. However, all arcs of the Cayley digraph have p-gons to their left, so I is on their left. Hence the arcs on the boundary have a consistent orientation. □

Corollary 6.3. *There is a two-way infinite directed Hamiltonian path in* Cay$([p, q]^+; R, S, T)$.

Proof. By interchanging p and q if necessary, we may assume $p > 3$. Then the theorem asserts there is a two-way infinite directed Hamiltonian path in Cay(R, T). Hence there is one in Cay(R, S, T). □

7. Negative results

Theorem 7.1. *There is no one-way infinite directed Hamiltonian path in* Cay$([p, 3]^+; R, S)$.

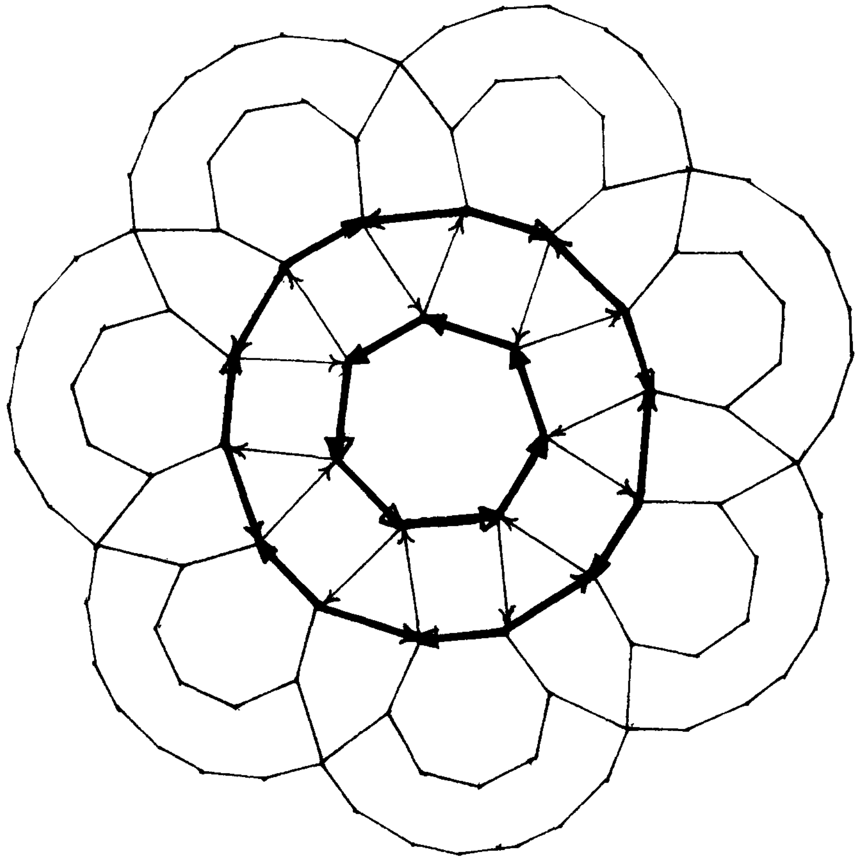


Fig. 19. A portion of Cay($[p,3]^+$; R, S).

Proof. Surrounding each p-gon is a concentric $2p$-gon whose arcs alternate in direction (see Fig. 19). Let one such $2p$-gon be $v_1 w_1 v_2 \ldots v_p w_p$ with arcs $\overrightarrow{v_1 w_1}, \overleftarrow{w_1 v_2}, \overrightarrow{v_2 w_2}, \ldots, \overleftarrow{w_p v_1}$. The only arcs leaving v_k are $\overrightarrow{v_k w_k}$ and $\overrightarrow{v_k w_{k-1}}$, so one of these two arcs is in the directed Hamiltonian path, On the other hand, there can never be two consecutive arcs of a $2p$-gon in the path, because then there would be more than one arc entering (or leaving) a single vertex. Therefore, the path contains (exactly) one of any two consecutive arcs on any $2p$-gon.

Let $\overrightarrow{ab}$ be the first arc in the one-way infinite directed Hamiltonian path. Note that a is on two different $2p$-gons, and one of these $2p$-gons (call it $\mathcal{O}$) does not contain $\overrightarrow{ab}$. Let x and y be the vertices adjacent to a on $\mathcal{O}$. Since a is the starting point of the path, neither $\overrightarrow{xa}$ nor $\overrightarrow{ya}$ is on the path. This contradicts the conclusion of the preceding paragraph. □

Theorem 7.2. *There is no one-way infinite directed Hamiltonian path in* Cay($[p,q]^+$; R, T).

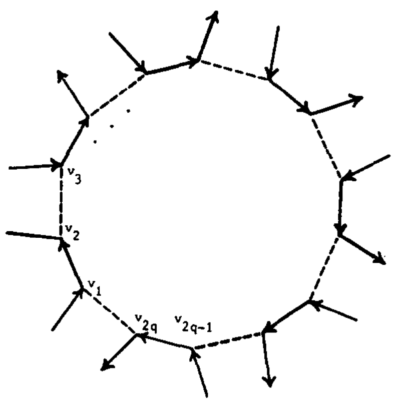


Fig. 20. A $2q$-gon in Cay($[p,q]^+$; R, T).

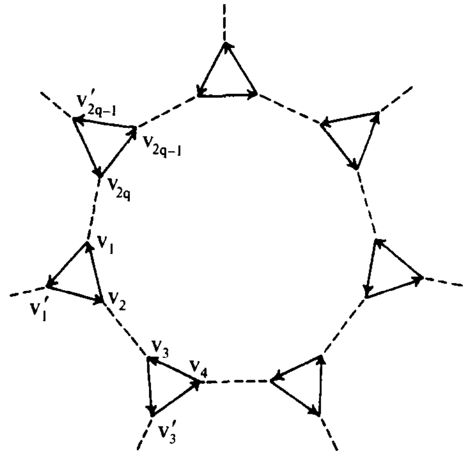


Fig. 21. A $2q$-gon in Cay($[3,q]^+$; R, T).

Proof. Let $v_1, \ldots, v_{2q}$ be a $2q$-gon as in Fig. 20, and assume without loss of generality that v_{2q} is the starting point of the one-way infinite directed Hamiltonian path. Then the only arc that can leave v_1 is $\overrightarrow{v_1 v_2}$. Similarly, the only arc that can leave v_3 is $\overrightarrow{v_3 v_4}$, and so on. Finally, the only arc that can leave v_{2q-1} is $\overrightarrow{v_{2q-1} v_{2q}}$. But v_{2q} is the starting point of the path, so this is a contradiction. □

Theorem 7.3. *There is no one-way infinite directed Hamiltonian path in* Cay($[p,q]^+$; R, S^{-1}).

Proof. Let a be the starting point of a one-way infinite directed Hamiltonian path. Then the path contains neither of the arcs $\overrightarrow{ba}$ and $\overrightarrow{ca}$ that terminate at a. So the path contains the only other arc out of each of b and c. But because $R^{-1}S^{-1} = SR$, these two arcs have the same terminal vertex (call it d), so the path has two arcs entering d. This is a contradiction. □

Theorem 7.4. *There is no two-way infinite directed Hamiltonian path in* Cay($[3,q]^+$; R, T).

Proof. Note first that the directed Hamiltonian path must traverse two consecutive arcs of each p-gon (i.e., triangle), except perhaps for the p-gon containing the starting vertex of the path. Now consider some $2q$-gon $v_1, v_2, \ldots, v_{2q}$, with v_1 and v_{2q} in different p-gons (as in Fig. 21). For each i, let v'_{2i+1} be the third vertex in the p-gon containing v_{2i-1} and v_{2i}. Assume the $2q$-gon is far from the starting vertex of the path, so the path traverses two arcs of $\triangle v_1 v'_1 v_2$. We may assume the path contains the arcs $\overrightarrow{v_1 v'_1}$ and $\overrightarrow{v'_1 v_2}$. From v_2 the path must go to v_3. The path traverses two arcs of

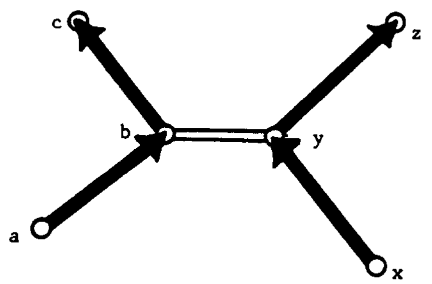


Fig. 22.

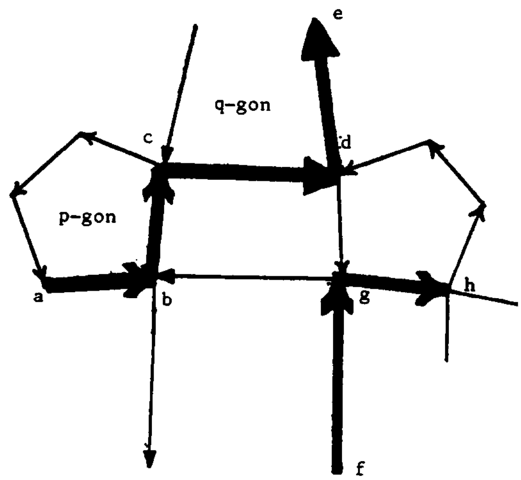


Fig. 23.

$\triangle v_3 v_3' v_4$ and goes from v_4 to v_5, and so on, until the path traverses two arcs of $\triangle v_{2q-1} v_{2q-1}' v_{2q}$ and goes from v_2 to v_1. This means the path traverses the cycle $\overrightarrow{v_1 v_1' v_2 v_3 \ldots v_{2q} v_1}$. This is impossible. □

Theorem 7.5. *There is no two-way infinite directed Hamiltonian path in* $\mathrm{Cay}([p^+, q]; R, T_p)$.

Proof. We argue by contradiction. Suppose that there is a two-way infinite directed Hamiltonian path. The path must include two consecutive arcs $\overrightarrow{ab}$ and $\overrightarrow{bc}$ of some p-gon. Let y be the third vertex adjacent to b. The directed Hamiltonian path must pass through y, so it contains the other two arcs $\overrightarrow{xy}$ and $\overrightarrow{yz}$ incident to y. (See Fig. 22.) Assume without loss of generality that the path passes through b before it reaches y. Then the directed Hamiltonian path has a subpath P from c to x, and $\mathscr{C} = \overrightarrow{xy} \cup \overrightarrow{yb} \cup \overrightarrow{bc} \cup P$ is a cycle whose removal disconnects the Cayley digraph. The two (infinite) ends of the directed Hamiltonian path are in different components $\mathrm{Cay}([p^+, q]) \backslash \mathscr{C}$ (one end terminates at a and the other begins at z), but only one component of $\mathrm{Cay}([p^+, q]) \backslash \mathscr{C}$ is infinite (Jordan curve theorem). This is a contradiction. □

Theorem 7.6. *There is no two-way infinite directed Hamiltonian path in* $\mathrm{Cay}([p, q]^+; R, S)$.

Proof. We begin by showing that no two-way infinite directed Hamiltonian path can contain two consecutive arcs of a p-gon, immediately followed by two consecutive arcs

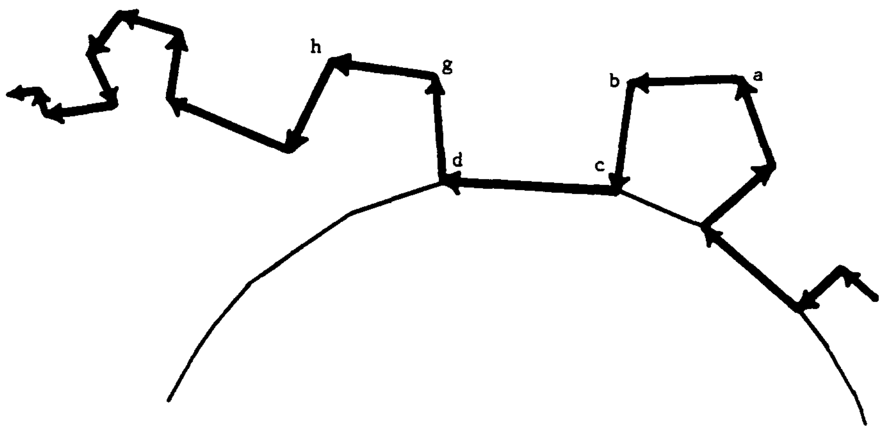


Fig. 24.

of a q-gon. To this end, suppose the path contains $\overrightarrow{ab}$, $\overrightarrow{bc}$, $\overrightarrow{cd}$, and $\overrightarrow{de}$ as in Fig. 23. Then it must also contain the path $\overrightarrow{fgh}$. It must, therefore, also contain either a path P from e to f or a path Q from h to a. In the first case, the cycle $P \cup \overrightarrow{fg} \cup \overrightarrow{gd} \cup \overrightarrow{de}$ encloses h or a. In the other case, the cycle $Q \cup \overrightarrow{ab} \cup \overrightarrow{bg} \cup \overrightarrow{gh}$ encloses e or f. In either case, the supposedly two-way infinite directed Hamiltonian path is limited to finite length in one direction. This is a contradiction, as in the proof of Theorem 7.5.

It is easy to see that a two-way infinite directed Hamiltonian path must traverse two consecutive sides of either some p-gon or some q-gon. Assume it is a p-gon. Because the path must eventually leave this p-gon, this means the path must have a subpath equivalent to $\overrightarrow{abcd}$ in Fig. 23. The preceding paragraph implies that the path must proceed from d to g. It must then go to h. But $\overrightarrow{dgh}$ is two sides of another p-gon. This establishes that the directed Hamiltonian path must traverse two (or more) arcs of a p-gon, then a single arc of some q-gon, then some arcs of a p-gon, again a single arc of some q-gon, some arcs of a p-gon,... The same argument shows that the other end of the path also never traverses two consecutive arcs of any q-gon. It follows that, as pictured in Fig. 24, the path can never cross the (hyperbolic) line through c and d. Thus the supposedly directed Hamiltonian path omits many vertices. □

The proof of Theorem 7.5 above uses a simple case of the Jordan curve theorem to show that no two-way infinite path can contain a subgraph as shown in Fig. 22. The following proof is based on a similar argument that the configurations depicted in Figs. 26 and 27 cannot occur in any one-way infinite path.

Theorem 7.7. *There is no one-way infinite directed Hamiltonian path in* $Cay([p^+, q]; R, T_p)$.

Proof. Assume the directed Hamiltonian path begins by traversing some arcs of a p-gon (suppose counterclockwise), and then leaves this p-gon at some vertex v. Let x be the vertex immediately preceding v on the path, and let z be the vertex immediately following v. To traverse x' and z' (see Fig. 25), the path must contain the arcs $\overrightarrow{x'c}$ and $\overrightarrow{z'e}$. After v, and before reaching either of x' and z', the path must traverse two consecutive sides of some p-gon.

If the first such p-gon is oriented counterclockwise, we get a configuration as in Fig. 26. Now, if z' occurs before w' in the path, then the portion of the path from z to z' plus arc $\overrightarrow{z'z}$ forms a circuit encircling either e or f. In either case, the path is limited to finite length. If w' occurs before z', then the portion of the path from w to w' plus arc $\overrightarrow{w'w}$ forms a circuit enclosing either e or f. Again, in either case, the path is limited to finite length.

On the other hand, if the first such p-gon is oriented counterclockwise, we get a configuration as in Fig. 27. Much as in the preceding paragraph, a portion of the path, plus either arc $\overrightarrow{x'x}$ or $\overrightarrow{y'y}$, forms a circuit enclosing either c or d. In either case, the path is limited to finite length.

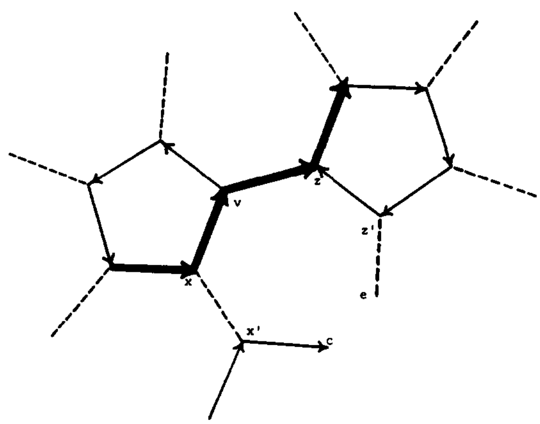


Fig. 25.

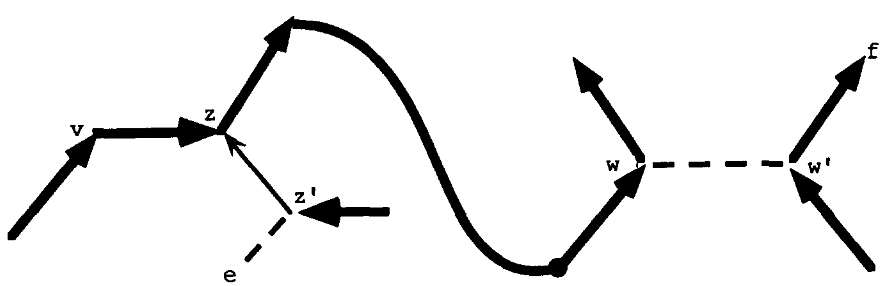


Fig. 26. At w, there are two consecutives edges of a counterclockwise p-gon.

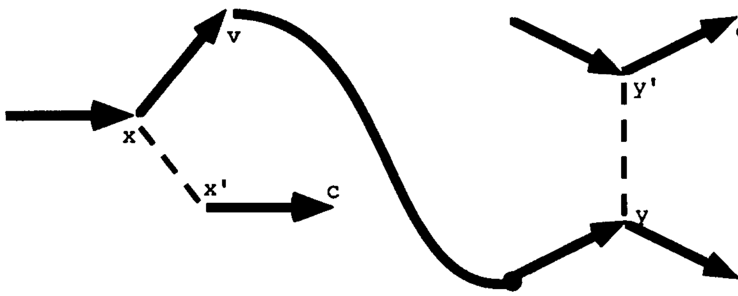


Fig. 27. At y, there are two consecutive edges of a counterclockwise p-gon.

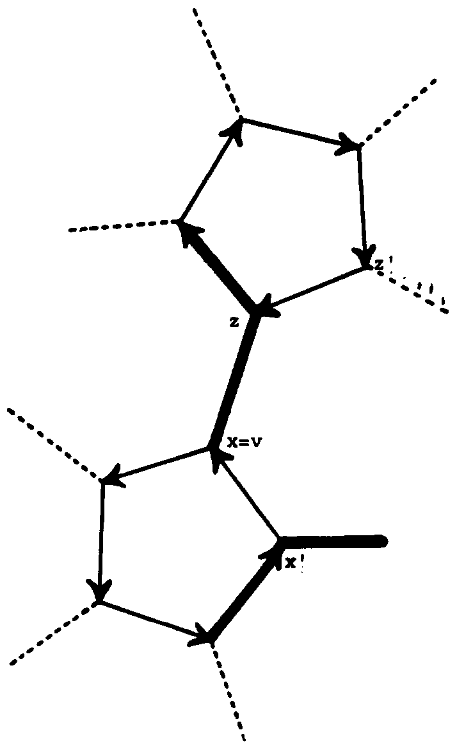


Fig. 28.

The case in which the first arc of the path is not an arc of a p-gon can be handled similarly. We simply must make another choice of x and x' (see Fig. 28). □

8. Hamiltonian paths in the graphs

Definition. A (one-way infinite or two-way infinite) *Hamiltonian path* in an infinite digraph G is a Hamiltonian path in the graph obtained from G by removing the orientation of its arcs. It is obvious that if a digraph has a directed Hamiltonian path, then it has a Hamiltonian path.

Theorem 8.1. *There are both one-way infinite and two-way infinite Hamiltonian paths in* $\mathrm{Cay}([p,q]^+; T_p, T_q, T_2)$ *and in* $\mathrm{Cay}([p,q]^+; R, S, T)$.

Proof. There are both one-way infinite and two-way infinite directed Hamiltonian paths in $\mathrm{Cay}([p,q])$ and in $\mathrm{Cay}(R,S,T)$ (see 5.4, 3.2, 4.7, and 6.3). □

Theorem 8.2. *There are both one-way infinite and two-way infinite Hamiltonian paths in* $\mathrm{Cay}([p,q]^+; R, S)$ *and in* $\mathrm{Cay}([p,q]^+; R, S^{-1})$.

Proof. Since $\mathrm{Cay}(R, S^{-1})$ is the same as $\mathrm{Cay}(R,S)$, except for the orientation of some arcs, a Hamiltonian path in either of these digraphs yields a Hamiltonian path in the other. Corollary 3.3 asserts the existence of a two-way infinite Hamiltonian path, and if $p,q > 3$, Theorem 4.1 asserts the existence of a one-way infinite Hamiltonian path. By symmetry, all that remains is to find a one-way infinite Hamiltonian path in $\mathrm{Cay}([3,q]^+; R, S)$. Start, as in the proof of Theorem 4.7, with a one-way infinite Hamiltonian path in the tessellation $\{q,3\}$ (see Fig. 7). Construct a one-way infinite Hamiltonian path in $\mathrm{Cay}([3,q]^+; R, S)$ by traversing the p-gons of $\mathrm{Cay}(R,S)$ in the order indicated by the path in the tessellation. □

Lemma 8.3. *There is a two-way infinite Hamiltonian path in the tessellation* $\{q,3\}$ *(for* $q \geqslant 7$*).*

Proof. It suffices to create a collection $\{Q_1, Q_2, \ldots\}$ of closed q-gons such that:

(1) Q_n shares a single edge with Q_{n-1}, and is disjoint from $Q_1, Q_2, \ldots, Q_{n-2}$; and

(2) every vertex of $\{q,p\}$ is in some q-gon Q_n.

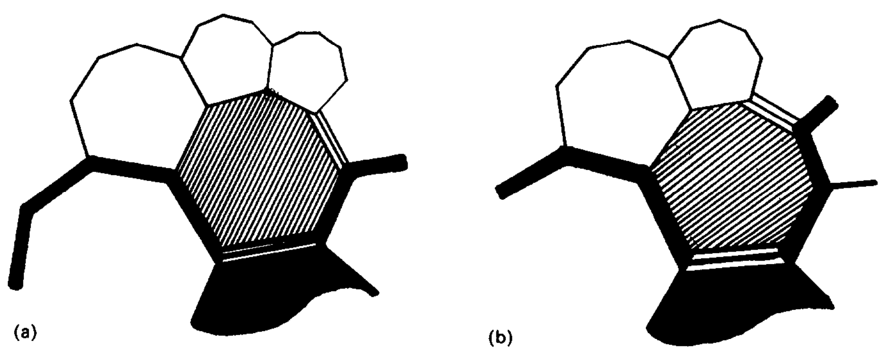


Fig. 29. Q_n is black; Q_{n+1} is shaded lightly. (a) 1 1 1 $\boxed{2\ 2}$ 1 transforms to $1\ 2\ 1\ 1_{\ldots}1\ 2\ 1_{\ldots}1$ $\boxed{2\ 2}$. (b) 1 1 1 $\boxed{2\ 2}$ 2 1 transforms to $1\ 2\ 1\ 1_{\ldots}1\ 2\ 1\ 1_{\ldots}1$ $\boxed{2\ 2}$.

(The boundary of the union of these q-gons is the desired two-way infinite Hamiltonian path in $\{q, 3\}$.) The construction of this collection is similar to the proof of Proposition 6.1. Begin by choosing a q-gon Q_1 and any edge of the q-gon. Call this the (first) *special* edge. For the induction step, let X_n be the union of all q-gons that intersect any of $Q_1, \ldots, Q_n$ anywhere other than at the (nth) special edge. Let Q_{n+1} be the q-gon that shares the special edge with Q_n. The $(n+1)$st special edge is the last (i.e., most clockwise) edge of Q_{n+1} on the boundary of $X_n \cup Q_{n+1}$. The possible transitions are pictured in Fig. 29. □

Theorems 8.4. *There are both one-way infinite and two-way infinite Hamiltonian paths in* $\mathrm{Cay}([p,q]^+; R, T)$ *and in* $\mathrm{Cay}([p^+,q]; R, T_p)$.

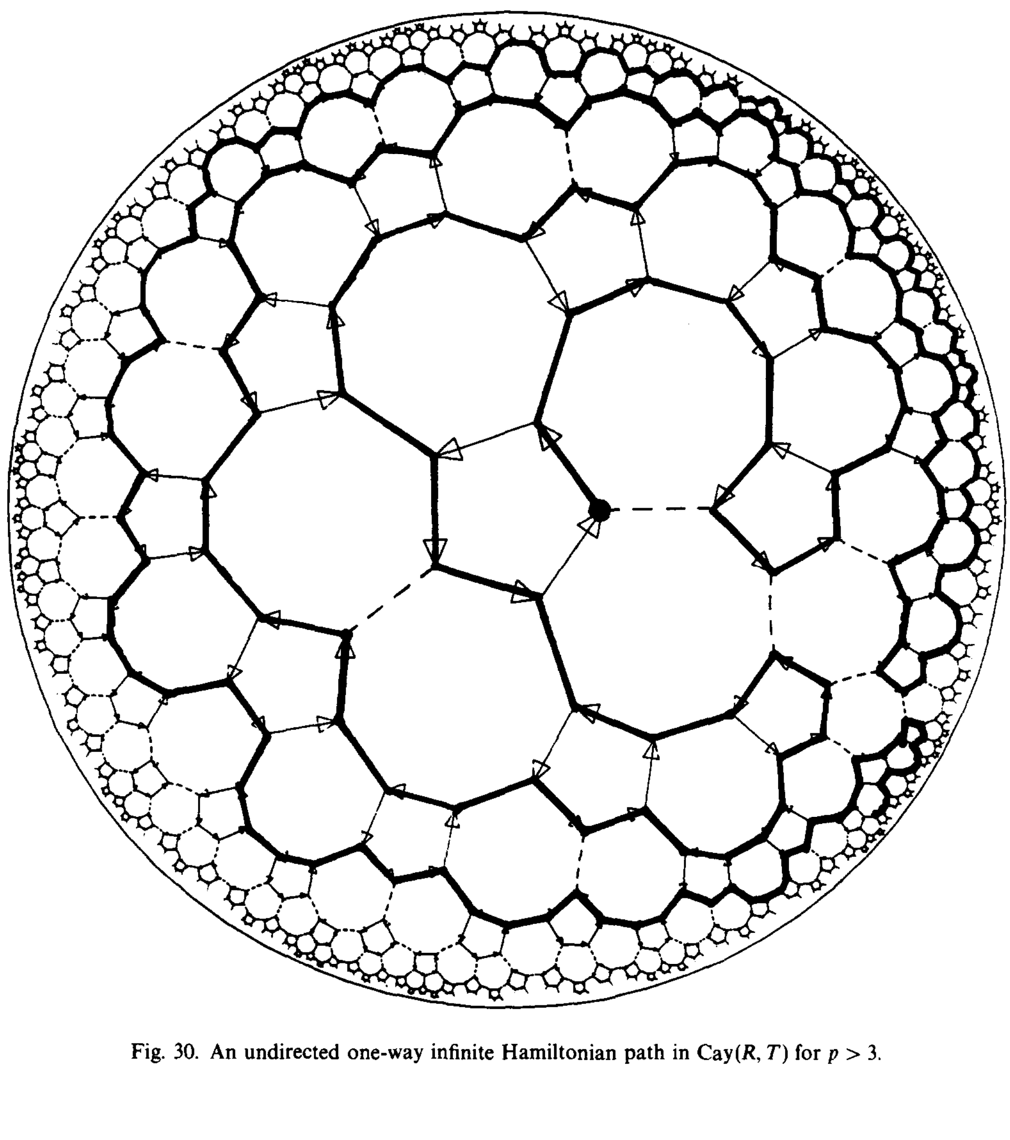


Fig. 30. An undirected one-way infinite Hamiltonian path in Cay(R, T) for $p > 3$.

Proof. Since Cay($[p^+, q]; R, T_p$) is the same as Cay($[p,q]^+; R, T$), except for the orientation of some arcs, any Hamiltonian path in the latter yields a Hamiltonian path in the former. We thus need only consider Cay($[p,q]^+; R, T$).

Begin by considering one-way infinite paths. If $p = 3$, a one-way infinite Hamiltonian path in Cay(R, T) can be constructed by traversing the p-gons of Cay(R, T) in the order indicated by a one-way infinite Hamiltonian path in the tessellation $\{q, 3\}$ (as in the proofs of Corollary 4.7 and Theorem 8.2). Now assume $p \geqslant 4$. We see that Cay(R, T) tessellates the hyperbolic plane into p-gons and $2q$-gons, and yields a natural filtration. The first filtration set is a single $2q$-gon, and the nth filtration set is the union of the $(n-1)$st filtration set with all p-gons or $2q$-gons adjacent to it. Because $p > 3$, one can show that every vertex is on the boundary of some filtration set, and these can be joined to form a one-way infinite Hamiltonian path (see Fig. 30).

Now consider two-way infinite paths. If $p > 3$, Theorem 6.2 asserts the existence of a Hamiltonian path in Cay(R, T), so we need only find a two-way infinite Hamiltonian path in Cay($[3,q]^+; R, T$). Much as in the proofs of 4.7 and 8.1, the p-gons of Cay(R, T) can be traversed in the order indicated by a two-way infinite Hamiltonian path in the tessellation $\{q, 3\}$. □

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